Fun in Podo’s Paddock: A New Way to Teach the Laws of Sign for Integer Multiplication and Division in Lower Grades

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Abstract

Many teachers find it difficult to explain the laws of sign to children. Mathematics is a science of patterns, so ${}^{+}2 \times {}^{+}3 = {}^{+}6$ suggests ${}^{-}2 \times {}^{-}3 = {}^{-}6$, yet this is not the case. Multiplicative pedagogies on the *Naturals*, such as; repeated addition, equal groups, arrays, and area models, fail on the *Integers*. You cannot add a number of objects a negative number of times, or have a negative number of groups, or a negative number of rows, or an area less than zero. Division has a repeated subtraction model and an equal shares model, yet without opaque sign laws, teachers cannot explain how to solve ${}^{+}6 \div {}^{-}3 = \square$. There aren’t any negative threes in positive six and you can’t divide six into negative three groups. So, new games are played, based on rules and writings of some of the most important mathematicians in history. Revealed in a two-act play featuring mathematicians as children, flaws in the teaching of elementary mathematics are found and fixed.

Introduction

In **Act 1**, Teacher challenges Dio (Diophantus), and Gerry (Cardano), to explain (10 – 2) × (10 – 3). After some confusion, they succeed. In **Act 2**, Teacher challenges Dio and Gerry to solve (2 – 10) × (3 – 10). They can’t. Yet, outside in Podo’s Paddock, Eucy (Euclid), Huiey (Liu Hui), and Brammy (Brahmagupta), laugh as they play a fun game that solves the problem! Teacher, not believing the problem can be solved, taunts the children with division involving a positive dividend and a negative divisor. Despite explaining neither numbers less than zero, nor laws of sign, Teacher is about to get a big surprise!

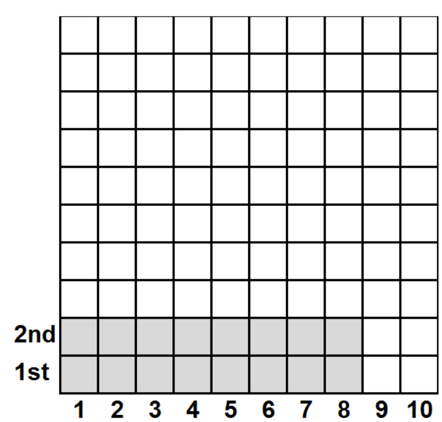
Act 1

TEACHER: Dio, place 7 lots of 8 blocks on a table.

DIO: Yes Teacher. First, I place 8 blocks on the first row. Then I place 8 blocks on the second row…

TEACHER: No, no, no! We may well build *up* and count *up*, yet Pythagoras’ multiplication table has numbers going *down*. Your multiplication table will never solve a problem like (10 – 2) × (10 – 3).

GERRY: Please explain this multiplication more.



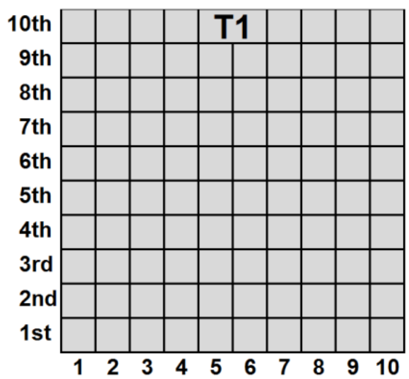
TEACHER: Good gracious! I thought you two children were meant to be smart!

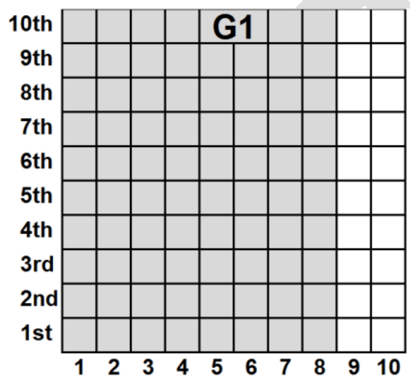
DIO: Wait, 8 blocks placed 7 times on a table is also said to be 8 blocks multiplied by 7. You have disguised 8 blocks as 10 – 2 blocks and disguised the 7 times they are to be placed together as 10 – 3.

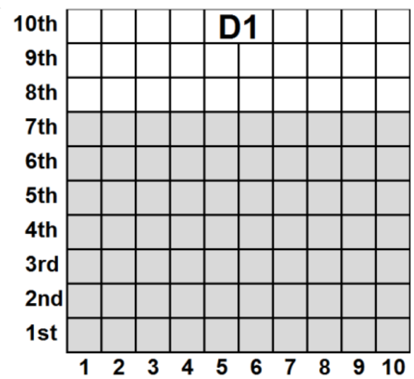
TEACHER: Indeed, we have (**10** – 2) × (**10** – 3). You must show me how you start with 100 blocks [**Fig. T1**] on your 10 × 10 table and end down with 56. All other children must take some toys and play outside until we are done!
[Eucy gets a straight-edge, Huiey gets a bucket and Brammy gets a spade.]

GERRY: I'm taking away 2 blocks added in excess from each row of 10. [**Fig. G1**]

DIO: I'm taken away the 8th, 9th and 10th rows of 10 added in excess. [**Fig. D1**].

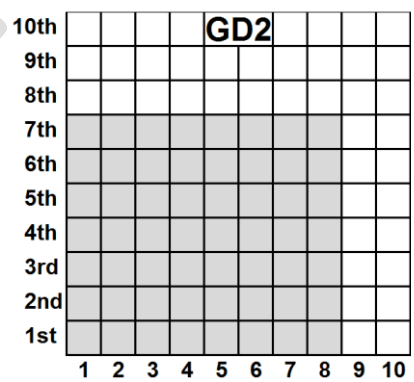






GERRY: So, I subtracted 20 blocks.

DIO: As I subtracted 30 blocks. So, on another table [**Fig. GD2**] we show 7 lots or rows of 8 blocks each, which is 56 blocks altogether!



[Gerry & Dio start to frown as they check their work.]

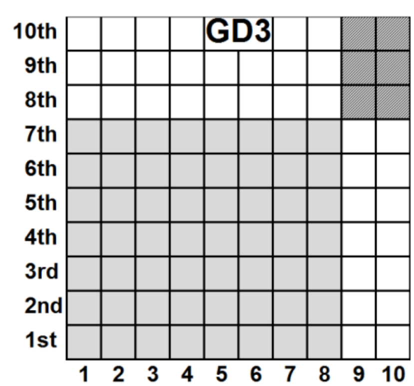
GERRY: From 100 blocks I took away 20, leaving 80.

DIO: While I took away 30, which leaves 50. We have 100 – 20 – 30 = 50, yet our calculation is meant to produce an answer of 56 blocks!

[Can you see where Dio and Gerry went wrong?]

GERRY: **You** went wrong! After I subtracted 20 blocks up Eastside, you couldn't have subtracted 3 full rows of 10 blocks from Northside as 6 blocks were already taken away!

DIO: No, **YOU** went wrong! After I subtracted 3 rows from Northside, you could not have subtracted 20 blocks from Eastside as I had taken away the 6 blocks **YOU** claim to have subtracted. [**Fig. GD3**]



GERRY: We can’t take away the same six blocks away two times. Our result with the blocks was correct [**Fig. GD2**], yet we must adjust our calculation by adding back the extra six blocks **YOU** took away by mistake. Then, our equation becomes 100 – 20 – 30 **+ 6** = 56.

NOTE *At last we see what is going on in the equation (10 – 2) × (10 – 3) = 56 which may be written in algebraic form* as $(\boldsymbol{a} - \boldsymbol{b}) \times (\boldsymbol{c} - \boldsymbol{d}) = \boldsymbol{ac} - \boldsymbol{bc} - \boldsymbol{ad} \underline{+ \boldsymbol{bd}}$. *Mathematicians DEFINE neg. × neg. = pos. to preserve the distributive property. Yet NONE of the numbers here are negative! In Arithmetica, (c. 250 CE), Diophantus wrote (in Greek) ‘A wanting multiplied by a wanting makes a forthcoming’ which, in the 16th Century, was translated into Latin as ‘Minus per minus multiplicatum, producit Plus.’ Read on...* ***Act 2*** *explains negative numbers!*

Act 2

TEACHER: Yes, yes. Minus by minus, a number gets plussed, the reason for this has now been discussed. Yet I challenge you to solve, not (10 – 2) × (10 – 3), but the problem of (2 – 10) × (3 – 10). If you are so smart, solve that!

GERRY: Teacher, it is one thing to start with 10 and take 2 away from it, yet to start with 2 and take 10 away from it is absurd! The first problem began with 100 blocks, from which I took away 20 while Dio took away 30.

DIO: The first step has 2 blocks placed 3 times, which produces 6 blocks. Neither 20 blocks nor 30 blocks can be subtracted from 6 blocks. It is impossible!

[From the classroom, the scene changes to kids playing in Podo’s Paddock.]

EUCY: I like earth measuring. My straight-edge can draw tables on the ground.

HUIEY: OK. The dual natures of Yin and Yang sum up the fundamentals of mathematics. Yet we need blocks for Yang and all I have is a bucket.

BRAMMY: With my spade we can dig holes for Yin and make blocks for Yang!

HUIEY: Yes! How funny is it that Teacher says negative numbers are less than zero! [The children fall about laughing.] In China, positives and negatives were being played with more than a thousand years before zero was used as a number! We just used numbers of red rods to count or measure Yang things and black rods to count of measure Yin things.

EUCY: What do you mean? Teacher says negative numbers are less than zero, yet never says less WHAT than zero!

BRAMMY: Negative numbers are less positive than zero and positive numbers are less negative than zero. A hole one meter deep is less high than ground level zero. Yet a block one meter high is less deep than ground level zero.

[Gerry has snuck out to join the gang.]

EUCY: Aha! Now I see the equal measure [sym-metry]. Numbers can be the same, while the units they describe are equal and opposite in nature.

GERRY: I can combine two lots of same things. Yet how do we join or combine equal and opposite things, like blocks and holes, or hot-rocks and ice-blocks, or East steps and West steps? Teacher told me to solve (2 – 10) × (3 – 10).

HUIEY: When two soldiers from opposite sides join in battle, they both die! So, whatever side has more soldiers wins! Should an Eastern army with 24 soldiers battle a Western army with 13 soldiers...

GERRY: Then the Eastern side wins with 11 soldiers remaining!

BRAMMY: In the West, opposites are more likely said to be: boy-girl, big-little, soft-hard, narrow-wide, fast-slow, clean-dirty, cheap-expensive, long-short, strong-weak, rough-smooth, sick-healthy, new-old and so on.

HUIEY: We had ideas like that in the East, yet when it came to mathematics, our ideas of opposites were just things that cancelled each other out.

BRAMMY: Teacher says zero is a number subtracted from itself written $\boldsymbol{n} - \boldsymbol{n} = 0$. Yet I say zero is the sum of equal Yin and equal Yang or two opposite numbers of equal size written $^{-}\boldsymbol{n}$ *&* $^{+}\boldsymbol{n} = 0$. *Take away* $^{-}\boldsymbol{n}$ from zero, or $^{-}\boldsymbol{n}$ *&* $^{+}\boldsymbol{n}$, and $^{+}\boldsymbol{n}$ remains. *Take away* $^{+}\boldsymbol{n}$ from zero, or $^{-}\boldsymbol{n}$ *&* $^{+}\boldsymbol{n}$, and $^{-}\boldsymbol{n}$ remains.

GERRY: So, that's why, if from 5 black rods Huiey takes away 7 black rods, his answer is 2 red rods. Once he has subtracted 5 black rods from 5 black rods he has nothing, which as zero, equals 2 red and 2 black. So, when he takes away the final 2 black rods, the 2 red rods remain! The Chinese are smart!

BRAMMY: China has negative numbers and positive numbers, yet not as moderns understand. Chinese children played with numbers of negatives and numbers of positives. Ask any western adult what negative seven minus negative four is and most say negative eleven. Yet every child knows 7 negatives minus 4 negatives is 3 negatives. Westerners are confused by negatives because they have the wrong definition of zero, which I made right.

EUCY: Westerners made numbers with Greek geometry, yet we did not have ideas of zero, positives and negatives. Teacher will call us soon. As there are four of us, I've drawn four tables. Now, we must multi-play (2 – 10) × (3 – 10).

HUIEY: Any math object, taken away from Brammy's zero becomes its opposite. So, 2 blocks take away 10 blocks may be thought of as 2 blocks take away (2 + 8) blocks. After we have taken away 2 blocks from 2 blocks, we have 0 blocks on ground level zero. So, we split 0 by rebuilding it as 8 holes and 8 blocks. Then when we take away 8 blocks, 8 holes remain!

BRAMMY: The multiplicand (2 blocks – 10 blocks) is our number of things to be multiplied and this equals 8 holes. The multiplier (3 – 10) tells us how many times our multiplicand is to be added or subtracted to zero. Obviously, when 3 additions battle with 10 subtractions, the subtractions win and 7 remain!

HUIEY: A Hole and a Block of equal size make Zero when the Block and Hole are combined [H & B = Z]. When a Block is taken away from Zero a Hole remains and when a Hole is taken away from Zero a Block remains.

Brammy: As we started with blocks, then (2 – 10) × (3 – 10) is as simple as 8 holes subtracted from ground level zero 7 times. To take away 8 holes you add 8 blocks. Adding 8 blocks 7 times gives you 56 blocks! Let me double check. From (2 – 10) × (3 – 10). we have (8 holes) taken or placed together (3 times minus 10 times). Eight holes placed 3 times means 24 holes are added to zero. Eight holes taken away 10 times means 80 blocks are added to zero. As 24 holes cancel out 24 of the 80 blocks, 56 blocks remain. It is correct!

GERRY: We mustn't solve problems with common sense. We have to use tables!

EUCY: We have four tables altogether, a bucket and a spade, so let's multi-play!

HUIEY: The rules will be as follows. Westside will be for hole-play and Eastside will be for block-play. Northside will keep track of the number of times holes or blocks are Added to zero and Southside will keep track of the number of times holes or blocks are Subtracted from zero. Because Teacher confuses subtraction with Yin or negative, and addition with Yang or positive, signs of Yin & Yang are superscripted while operations remain standard symbols.

Brammy: For 2 bumps and 10 holes we write (**$^{+}$2** & $^{-}$10). The multiplier telling us to either add or subtract on ground level zero appears as (**0 + 3** & 0 – 10). So, we first distribute by adding 2 blocks 3 times to zero in **Q1**.

HUIEY: Then we play (**$^{+}$2** & $^{-}$10) × (0 + 3 & **0 – 10**), or 2 blocks subtracted 10 times. This can only be done in **Q4**. Blocks taken away from zero make holes.

BRAMMY: Then we play ($^{+}$2 & **$^{-}$10**) × (**0 + 3** & 0 – 10) or 10 holes added 3 times to zero. This can only be done in **Q2.**

EUCY: I see what you are doing. We then play ($^{+}$2 & **$^{-}$10**) × (0 + 3 & **0 – 10**) which is 10 holes subtracted 10 times from zero. This can only be done in **Q3.** We now see four partial products which may be combined into one number!

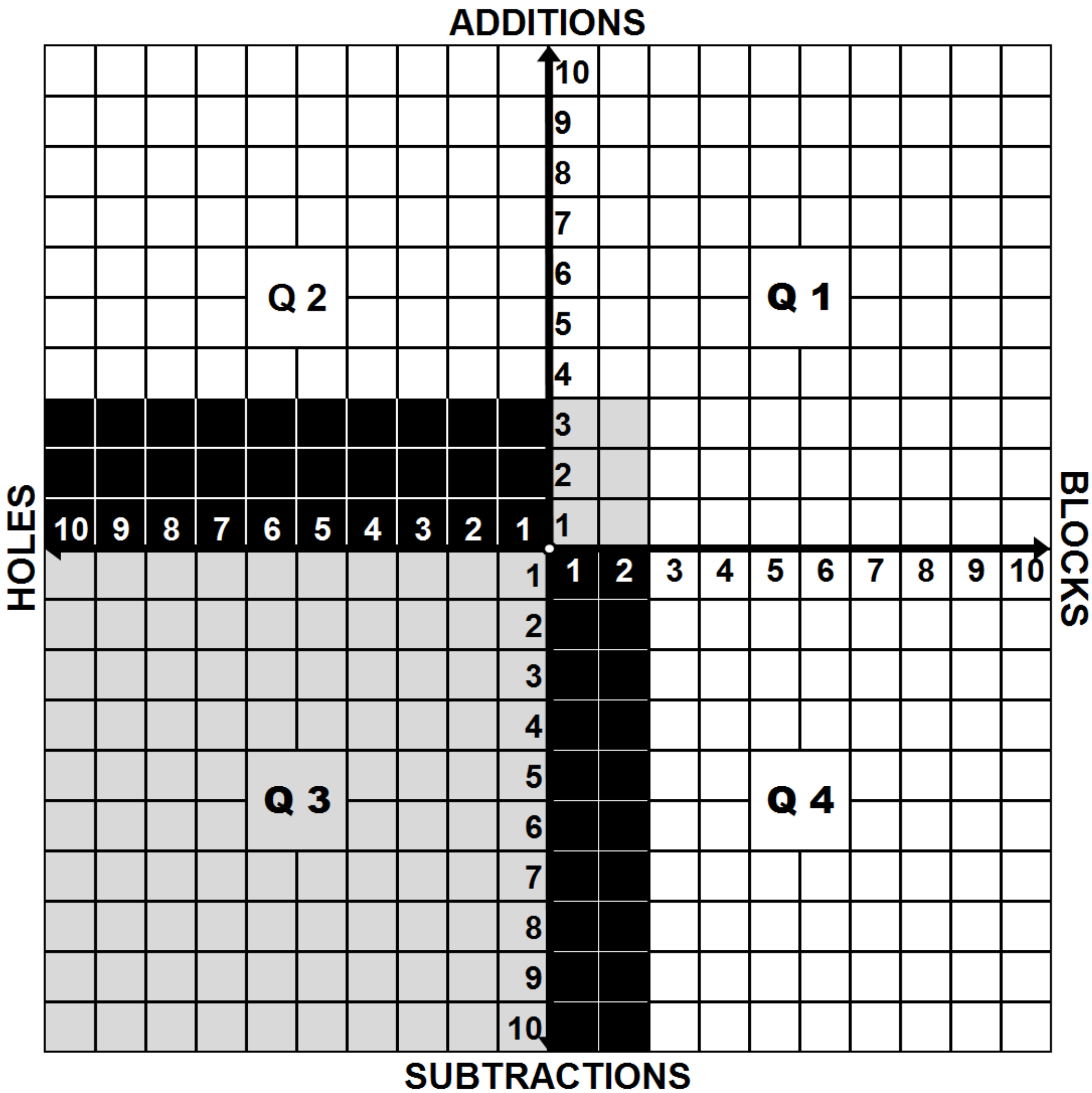
BRAMMY: So, 6 blocks in **Q1** and 100 blocks in **Q3** is 106 blocks.

HUIEY: And 20 holes in **Q4** and 30 holes in **Q2** is 50 holes. When 106 blocks battle with 50 holes, the blocks will win and 56 blocks will remain in **Q3**.

BRAMMY: We must fetch Dio so he can behold the wonder of negative area and positive area for the first time. We must fetch Teacher, so he can see at last, how it is impossible to have quantities that are less than zero. The impossible problem of (2 – 10) × (3 – 10) is a simple area on Podo's paddock. What fun!

$^{+}$2 × (0 + 3) 2 Blocks Added 3 Times onto Zero 6 Blocks in **Q1**	$^{+}$2 × (0 – 10) 2 Blocks Subtracted 10 Times from Zero 20 Holes in **Q4**	$^{-}$10 × (0 + 3) 10 Holes Added 3 Times onto Zero 30 Holes in **Q2**	$^{-}$10 × (0 – 10) 10 Holes Subtracted 10 Times from Zero 100 Blocks in **Q3**

(2 – 10) × (3 – 10) Played on Podo's Paddock



HUIEY: As with the ebb and flow of Yin and Yang, what was Product, becomes Dividend. What were Multiplicand and Multiplier become Quotient and Divisor.

BRAMMY: Positive products and dividends live in **Q1** by addition or **Q3** by subtraction. Negative products and dividends live in **Q2** by addition or **Q4** by subtraction. A positive dividend with a subtractive divisor requires negatives.

TEACHER: Stop your laughing! Playtime is over! Here's trouble! What's 56 ÷ $^{-}$7?

ALL: **EIGHT HOLES!**

THE END.

NOTE *An extended free PDF version of this play, with references and feedback will be available online at* www.jonathancrabtree.com *from October 2017.*